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Very fast approximate reconstruction of MR images

P.A. Angelidis *

BIOTRAST S.A., Thessaloniki, Greece

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Abstract

The ultra fast Fourier transform (UFFT) provides the means for a very fast computation of a magnetic resonance (MR) image, because it is implemented using only additions and no multiplications at all. It achieves this by approximating the complex exponential functions involved in the Fourier transform (FT) sum with computationally simpler periodic functions. This approximation introduces erroneous spectrum peaks of small magnitude. We examine the performance of this transform in some typical MRI signals. The results show that this transform can very quickly provide an MR image. It is proposed to be used as a replacement of the classically used FFT whenever a fast general overview of an image is required.

Keywords: MR image; Fourier transform; Periodic functions

1. Introduction

In the last years a number of fast methods have been proposed for magnetic resonance imaging (MRI) reconstruction. These can be classified as single-shot or multiple-shot techniques. The first group contains techniques such as echo planar imaging and its variations (MBEST being the most popular among them), which produce the necessary data to reconstruct the image within periods of tenths of milliseconds [1]. The second group contains low flip angle techniques, such as Snapshot FLASH, GRASS and various SSFP techniques, which produce the necessary data to reconstruct the image within periods of hundreds of milliseconds [2]. As a result of these short acquisition times, the reconstruction time has become an important fraction of the total time necessary to produce an image.

* Present address: Stefanou Nouka 20, Harilaou, GR-54250 Thessaloniki, Greece. Tel.: +30 93 860993; fax: +30 31 498280; e-mail: pantelis@biotrast.gr

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Furthermore, if k-space[1] information is transformed and displayed in real time, it will lead to clinical MR investigations of reduced duration. In other words a typical MR examination will last less, because the operator, having a general overview of the final image, will be able to react sooner, for example by modifying some of the imaging parameters. To achieve this kind of on-the-fly information, a very fast reconstruction algorithm must be available.

Usually, the reconstruction algorithm consists of a series of digital Fourier transforms (DFTs) applied initially horizontally (readout axes) and then vertically (phase encoding axes). The DFT sum of a two dimensional (2D) sequence $x[n, m]$ consisting of $N \otimes M$ (N times M) samples (n, m are integers, $0 \leq n < N$, $0 \leq m < M$) is defined by:

$$F(\omega_x, \omega_y) = \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} f(n\omega_x, m\omega_y) x[n, m],$$

$$0 \leq \omega_x, \omega_y < 1 \qquad (1)$$

where with $f(\omega_x, \omega_y)$ we denote the 2D complex exponential function (cef) $e^{j2\pi(\omega_x + \omega_y)}$. Usually, the DFT is computed at N and M equidistant frequency points along each direction, respectively, thus resulting to the following, more familiar definition [3]:

$$X[k_x, k_y] = \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} W_N^{nk_x} W_M^{mk_y} x[n], \quad k_x, k_y \in Z,$$

$$0 \leq k_x < N, \quad 0 \leq k_y < M \qquad (2)$$

where

$$W_N = e^{-j2\pi/N}$$

and

$$\omega_x = k_x/N, \qquad \omega_y = k_y/M \qquad (3)$$

i.e. ω_x is discretised in N and ω_y in M equidistant points.

The 2D sequence $X[k_x, k_y]$ constitutes a perfect sampling sequence of the FT of the MRI signal, i.e. the MR image, as long as the Nyquist criterion is satisfied during the acquisition.

Almost always the DFT is computed using the well-known FFT algorithm, which reduces the computational load from $O(N^2M^2)$ to $O(NM \log_2(N+M))$.[2] In order to further reduce the reconstruction time one can approximate the cef $f(\omega_x, \omega_y)$ by a more efficient, from a computational point of view, periodic function. This way the computational effort will be less, but at the cost of introducing artifacts in the final result. However, in cases where a very fast general overview of the image is desired, such an approach could be useful.

The performance of an algorithm which uses the computationally most efficient approximation function was examined here. This, in both dimensions periodic function takes only four values, namely 1, -1, j and $-j$. The reduction of the computational load is achieved at the cost of the introduction of aliasing artifacts as we will soon demonstrate. Using this approximation the multiplications in the DFT defining Eq. (2) become trivial; they completely vanish or become a mere sign change or a real-to-imaginary swap. The one dimensional (1D) version of the algorithm based on this four-values approximation function of the cef has been successfully applied in the processing of the MRS signals [4].

2. Analysis

Let

$$G(\omega_x, \omega_y) = \sum_{n=0}^{N} g(n\omega_x, m\omega_y) x[n, m] \qquad (4)$$

be the transform computed when approximating the cef $f(\omega_x, \omega_y)$ by $g(\omega_x, \omega_y)$. Since $g(\omega_x, \omega_y)$ is a periodic function in both dimensions, it can be expanded as a Fourier series [5] and be written as:

[1] In MRI literature the sampling space is known as k-space. Therefore, with k-space information we refer to the raw data gathered during excitation and acquisition in every MRI sequence. This data are then processed, usually using Fourier transform techniques, to produce the MRI.

[2] With the term computational load we refer to the strict computation theory definition of an addition and a multiplication. The sign $O(x)$ is used in computation theory to characterise an algorithm that performs x additions and/or multiplications.

$$g(\omega_x, \omega_y) = \sum_k \sum_l a_{k,l}\, e^{j2\pi(k\omega_x + l\omega_y)} \quad (5)$$

where the parameters $\alpha_{k,l}$ are the Fourier series coefficients.

Inserting Eq. (5) to Eq. (4) one gets:

$$G(\omega_x, \omega_y) = \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} \sum_k \sum_l a_{k,l}\, e^{j2\pi(nk\omega_x + ml\omega_y)} x[n,m] \quad (6)$$

Switching the order of summation and taking into account Eq. (1) one finds:

$$G(\omega_x, \omega_y) = \sum_k \sum_l a_{k,l} F(k\omega_x + l\omega_y) \quad (7)$$

Thus, $G(\omega_x, \omega_y)$ is a good approximation of the DFT of the sequence $x[n, m]$, $F(\omega_x, \omega_y)$, as long as $\alpha_{0,0}$ is close to one and all $\alpha_{k,l}$ for all the other values of the pair (k, l) are near zero. In fact, for $F(\omega_x, \omega_y)$ holds exactly that

$$\alpha_{k,l} = \begin{cases} 1 & \text{for } k = l = 0 \\ 0 & \text{otherwise} \end{cases}$$

In other words, the FT, not being an approximation, is represented by zero error terms $\alpha_{k,l}$-for k not equal to l.

The error terms $a_{k,l}F(k\omega_x, l\omega_y)$ are easily identified as aliased harmonics of the basic frequency pair (ω_x, ω_y). Following Eq. (3), we set the four-value approximation function equal to:

$$g_{4,4}(\omega) = e^{j2\pi(k/4 + l/4)} = e^{j\pi(k + l)/2} \quad (8)$$

Since $0 \leq k, l < 4$, it is obvious that the function $g_{4,4}(\omega)$ takes only the four values 1, -1, j and $-j$. It turns out, easily, that:

$$a_{k,l} = \begin{cases} \dfrac{(-1)^{q^l}}{k} \dfrac{(-1)^r}{l} \sin c^2(\pi/N) \\ 0 \text{ otherwise} \end{cases} \quad (9)$$

$$\text{if } k = 4q + 1 \quad \text{and} \quad l = 4r + 1, \quad q, r \in Z$$

Thus, Eq. (7) becomes:

$$G(\omega_x, \omega_y) = \frac{8}{\pi^2}\Big(F(\omega_x, \omega_y) + \frac{1}{3}(F(-3\omega_x, \omega_y) + F(\omega_x, -3\omega_y)) - \frac{1}{5}(F(5\omega_x, \omega_y) + F(\omega_x, 5\omega_y)) + \frac{1}{9} F(3\omega_x, 3\omega_y) \ldots \Big) \quad (10)$$

Eq. (10) shows that the transform in Eq. (7) for the four-values approximation function $g_{4,4}(\omega)$ in Eq. (8) yields a scaled version of the FT of the sequence $x[n]$, plus erroneous harmonics which become less important as their frequencies increase. Due to its speed, I call this transform: the ultra fast Fourier transform (UFFT). In the Appendix A a simple C routine which computes the 1D UFFT is given.

Two MR algorithms based on this transform are due. In the first the 2D UFFT is applied instead of the conventional FFT in both dimensions. In the second, a hybrid one, the FFT is applied along the readout direction and the UFFT along the phase-encoding one. Since most of the calculation delay is due to calculating the FT along the phase-encoding direction, time saving in this case is of the same order, but a much improved result is obtained (see discussion in the next section).

3. Application

The algorithm was applied to a standard spin-warp gradient-echo sequence of an image consisting of two phantom tubes, one filled with tap-water and the other with olive-oil. Data were acquired at a CISCO 4.7T machine, located at NMR Imaging Facility Centre, Queen Mary's College, London, UK. One hundred and twenty eight echoes were acquired, each one sampled at 128 points in time. The FFT image is shown in Fig. 1. In Fig. 3 the UFFT image is shown. Observe, that the basic structure has been revealed. In addition, erroneous harmonic terms can be seen, surrounding the real image. A much

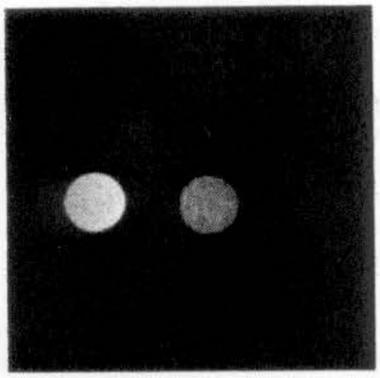

Fig. 1. The test phantom image reconstructed using a standard FFT algorithm.

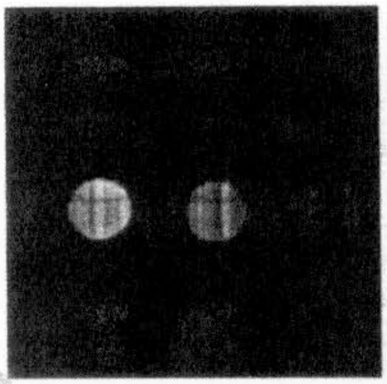

Fig. 3. The image of Fig. 1 reconstructed using the UFFT algorithm.

improved result is obtained if the FFT is applied along the readout direction and the UFFT along the phase-encoding one (hybrid transform), as is shown in Fig. 2. This time the erroneous terms are less important and appear only along the vertical direction.

A 256 × 128 gradient-echo image of a spinal cord in vitro is shown next (Fig. 4). The hybrid transform image is shown in Fig. 5 and the UFFT image in Fig. 6. Observe again that a general overview of the original image is quite possible using one of the last two images. The erroneous terms have the same occurrence as before.

Table 1 shows the computing time for each of the six images discussed above. Note that the difference between the hybrid and the full UFFT is not critical in multiple-shot MR sequences (which include standard spin-warp reconstruction techniques such as proton density, $T2$, $T1$ and low flip angle techniques such as FLASH, GRASS and SSFP), because the transformation along the read-out direction is carried out during sampling. However, the difference is important in one-shot (rapid) techniques (such as echo-planar and MBEST among others) in which all the data are sampled after a single excitation.

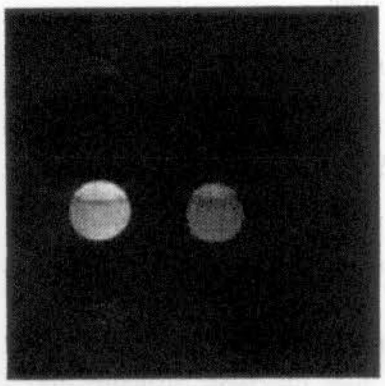

Fig. 2. The image of Fig. 1 reconstructed using a hybrid approach.

4. Conclusion

We developed an algorithm, which we call UFFT for computing an MR image very quickly, using only additions. This is achieved by approximating the complex exponential functions involved in the computation of the FT by periodic functions which take only the four values 1, -1, j and $-j$.

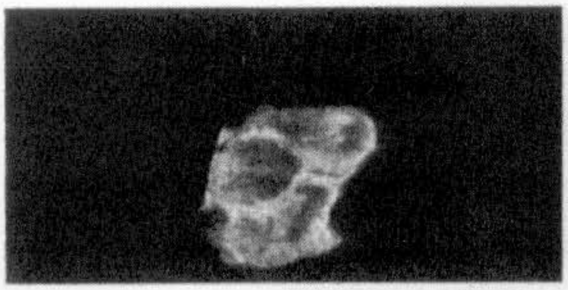

Fig. 4. The spinal cord image reconstructed using a standard FFT algorithm.

Fig. 5. The image of Fig. 4 reconstructed using a hybrid approach.

The analysis showed that this approximation introduces error terms, which are aliased harmonics of the actual frequency components present. The results showed that this fast transform may be used in MRI applications as a replacement of the FFT when a fast overview of the basic image structure is desired.

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Appendix A

A C routine that computes the UFFT is presented here.

comadd(complex,complex) is a function which adds two complex numbers, whereas comjadd(complex,complex) is a function which adds two complex numbers prior to shifting real and imaginary parts of the second.

Fig. 6. The image of Fig. 4 reconstructed using the UFFT algorithm.

Table 1

Image	Time (ms)	Difference to FFT (ms)
Fig. 1 (FFT)	439	0
Fig. 2 (hybrid)	302	137
Fig. 3 (UFFT)	165	274
Fig. 4 (FFT)	658.5	0
Fig. 5 (hybrid)	453	205.5
Fig. 6 (UFFT)	247.5	411

```
uf_dft(x, y, N)
  complex x,y;
  unsigned N;
{
  unsigned i,k;
  int f;
  double w,ws,ww;
  complex sum;


  ws = 1./(double)N;
  w = 0.;
  for(i = 0;i < N;i + + ) {
    sum = compeq(0.,0.);
    ww = 0
    for(k = 0;k < N;k + + ) {
      f = (int)(ww*8.);
      if(f > 7) f- = 8;
      switch(f) {
        case − 1:
                0: sum = comadd(sum,x[k]);
            break;
        case   1:
                2: sum = comjadd(sum,x[k]);
            break;
        case   3:
                4: sum = comadd(sum,-x[k]);
            break;
        case   5:
                6: sum = comjadd(sum,-x[k]);
            break;
```

```
ww + = w;
}
y[i] = sum;
w + = ws;
}
}
```

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