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Time–Frequency Representation of Damped Sinusoids Using the Zak Transform

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The Zak transform is one of the various time–frequency representations that are used in the field of signal analysis. We examine its performance for the type of signals encountered in NMR spectroscopy. A discrete version of the Zak transform is derived and typical examples are presented. The results are compared with those obtained from the Wigner distribution and standard techniques for the quantitative analysis of FIDs.

Time–frequency representations offer a dynamic tool for describing time-varying signals and systems. The advantages of these representations, as opposed to the time-invariant Fourier analysis, have made them very popular for processing various types of signals such as speech, music, and chirp signals. The importance of applying time-varying frequency estimators to NMR signals was realized for the first time in (*1*). There, one of these representations, the Wigner transform (WT), was proposed for the quantitative analysis of FIDs.

In this work, we examine the performance of another time–frequency representation, the Zak transform (ZT), which appears to exhibit some advantages over the WT. Historically, this transform was used for the first time in 1967 by Zak (*2*) for solving certain linear equations.[1] Several years later it was used in communication and transmission problems (*3*).

DEFINITION OF THE ZAK TRANSFORM

Let $x(t)$ denote a complex-valued time-continuous signal and $x[kT]$ denote a sequence produced by sampling $x(t)$ at a sampling rate of $1/T\,(-\infty < k < \infty)$. The ZT of $x[kT]$ is defined by

$$Z_x(t,f) = T^{1/2} \sum_{k=-\infty}^{\infty} x(t+kT)\exp(-j2\pi kfT),\qquad 0 \leq t \leq T,\ 0 \leq f \leq (1/T),\qquad [1]$$

where j denotes the imaginary unit. By equating T to unity and using normalized time and frequency variables (denoted by τ and ν, respectively), Eq. [1] becomes

$$Z_x(\tau,\nu) = \sum_{k=-\infty}^{\infty} x[\tau+k]\exp(-j2\pi k\nu),\qquad -\infty < \tau, \nu < \infty.\qquad [2]$$

Most of the basic properties of the Zak transform may be found in (*3*). We give here the formula that relates the ZT with the Wigner distribution (WD):

$$|Z_x(\tau,\nu)|^2 = 1/2 \sum_{n,m=-\infty}^{\infty} (-1)^{nm}\mathrm{WD}_{x,x}[\tau+(n/2),\ \nu+(m/2)],\qquad -\infty < \tau,\nu < \infty.\qquad [3]$$

The above formula will be used later to compare the results of the application of each of the two representations to the NMR signal.

REPRESENTATION OF DAMPED SINUSOIDS

We begin by examining a sequence $x[t]$ composed of a sum of M undamped sinusoids with amplitudes A_n, frequencies $\omega_n/2\pi$, and phases ϕ_n:

$$x[t] = \sum_{n=1}^{M} A_n\exp(j[\omega_n t+\phi_n]).\qquad [4]$$

From [2] and [4] one finds

$$Z_x(\tau,\nu) = \sum_{k=-\infty}^{\infty}\sum_{n=1}^{M} A_n\exp(j[\omega_n(\tau+k)+\phi_n])\exp(-jk\nu).\qquad [5]$$

[1] In particular, Zak used this transform (which he calls the *k–q* representation) to bring the Schrödinger equation for an electron subjected to a periodic potential in a constant magnetic field into a more manageable form. It is remarkable that both the Wigner distribution and the Zak transform were originally introduced in quantum mechanics.

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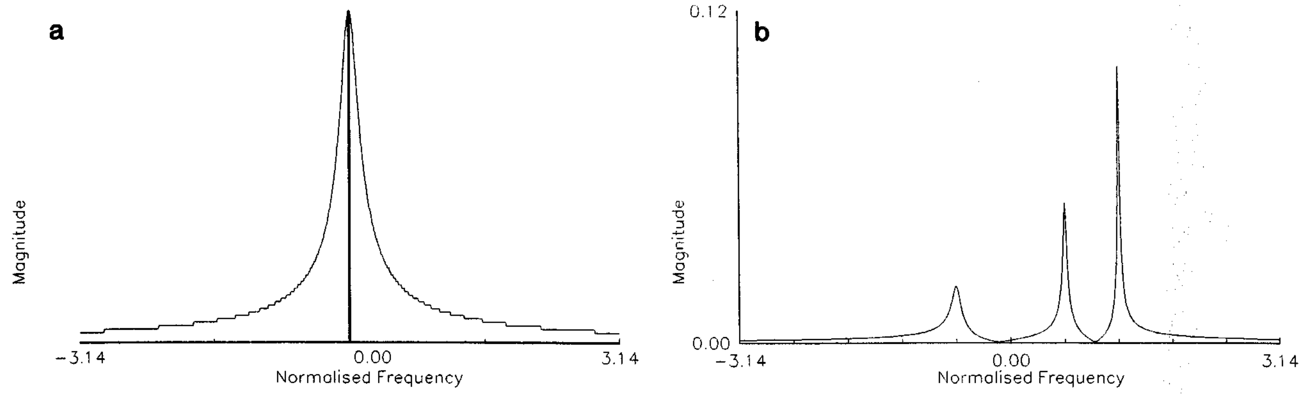

FIG. 1. (a) The magnitude of the ZT of a damping exponential (Eq. [17]). (b) The magnitude of the ZT of three damping exponentials with normalized frequencies of -0.4π, 0.1π, and 0.2π and damping factors of 50, 20, and 10 (normalized units), respectively. Observe that the region of support for each sinusoid is linearly dependent on its damping factor as Eq. [19] shows. In this case, condition [20] holds and the ZT is given by Eq. [21].

Reversing the order of the sums, after some simple manipulations, one gets

$$Z_x(\tau, \nu) = \sum_{n=1}^{M} A_n \exp(j\omega_n \tau)\exp(j\phi_n) \times \sum_{k=-\infty}^{\infty} \exp(j\omega_n k)\exp(-jk\nu). \quad [6]$$

The second sum in the above equation can be identified as the Fourier transform of the sequence

$$x_n[t] = \exp(j\omega_n t), \quad [7]$$

which is known to be

$$F_n(\nu) = \delta(\nu - \omega_n), \quad [8]$$

where δ denotes the Dirac impulse function. From [6] and [8], one arrives at

$$Z_x(\tau, \nu) = \sum_{n=1}^{M} A_n \exp(j\omega_n \tau)\exp(j\phi_n) F_n(\nu) \quad [9a]$$

or

$$Z_x(\tau) = \begin{cases} A_n \exp(j\omega_n \tau)\exp(j\phi_n) & \text{for } \nu = \omega_n \\ 0 & \text{elsewhere.} \end{cases} \quad [9b]$$

Finally,

$$|Z_x(\tau)| = \sum_{n=1}^{M} A_n \delta(\nu - \omega_n). \quad [10]$$

Thus, the magnitude of the ZT of a sum of sinusoids consists of peaks located at the frequencies of these sinusoids.

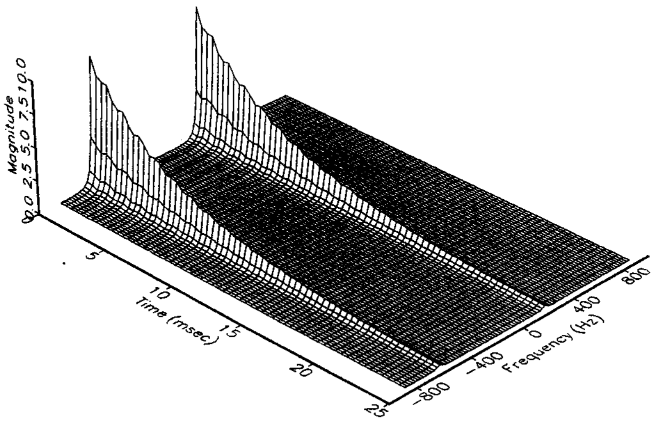

FIG. 2. The magnitude of the DZT of the sum of the two sinusoids.

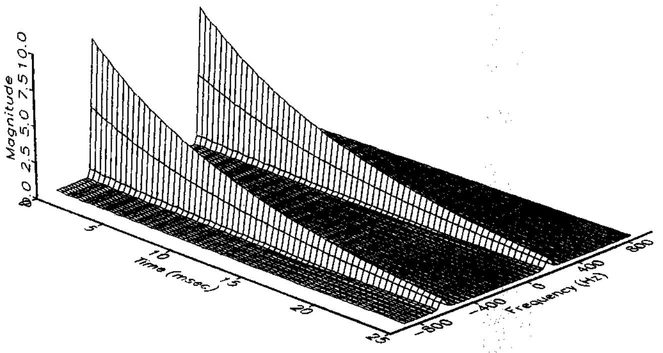

FIG. 3. The magnitude of the WDZT of the sum of the two sinusoids using a Hamming window.

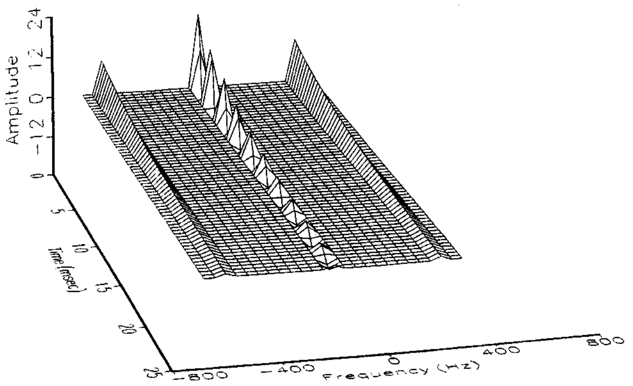

FIG. 4. The DWT of the sum of the two sinusoids.

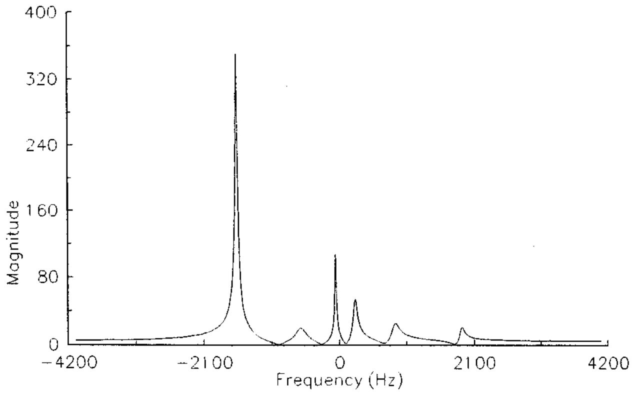

FIG. 5. Spectrum of a simulated FID without noise.

The WT of $x[t]$ is given by (*1*)

$$\mathrm{WD}(\tau, \nu) = \sum_{n=1}^{M} A_n^2 \delta(\nu - \omega_n) + 2 \sum_{n=1}^{M-1} \sum_{k=n+1}^{M} A_n A_k \cos[(\omega_n - \omega_k)\tau + (\phi_n - \phi_k)] \delta\left(\nu - \frac{\omega_n + \omega_k}{2}\right). \quad [11]$$

Note that Eq. [10] may be also derived by inserting [11] into [3]. Comparing [10] and [9], one sees that the so-called cross terms that appear midway between every two frequency components in the WT do not appear in the ZT. This observation demonstrated one major advantage of the ZT over the WD.

We focus now on the more general sequence

$$x[t] = \sum_{n=1}^{M} A_n \exp(-\alpha_n t)\exp(j[\omega_n t + \phi_n]), \quad t \geq 0, \quad [12]$$

where α_n denotes the damping factor of the nth sinusoid (the T_2^* in NMR terminology) and the rest of the symbols are as in [4].

TABLE 1
Parameters of the Simulated Phosphorus FID

Peak	Frequency (Hz)	Time constant (ms)	Amplitude	Phase (°)
Reference	−1590	11	32,000	55
P_i	−600	2	10,000	83
PCr	−60	20	6,000	98.5
γ	240	6	9,000	107.5
α	860	3	8,000	122.5
β	1900	5	4,000	153

From [2] and [12], one finds

$$Z_x(\tau, \nu) = \sum_{k=-\infty}^{\infty} \sum_{n=1}^{M} A_n \exp[-\alpha_n(\tau + k)] \times \exp(j[\omega_n(\tau + k) + \phi_n])\exp(-jk\nu). \quad [13]$$

Reversing the order of the sums, after some simple manipulations one gets

$$Z_x(\tau, \nu) = \sum_{n=1}^{M} A_n \exp(-\alpha_n \tau)\exp(j\omega_n \tau)\exp(j\phi_n) \times \sum_{k=-\infty}^{\infty} \exp(-\alpha_n k)\exp(j\omega_n k)\exp(-jk\nu). \quad [14]$$

The second sum in the above equation can be identified as the Fourier transform of the sequence

$$x_n[t] = \exp(-\alpha_n t)\exp(j\omega_n t), \quad t \geq 0. \quad [15]$$

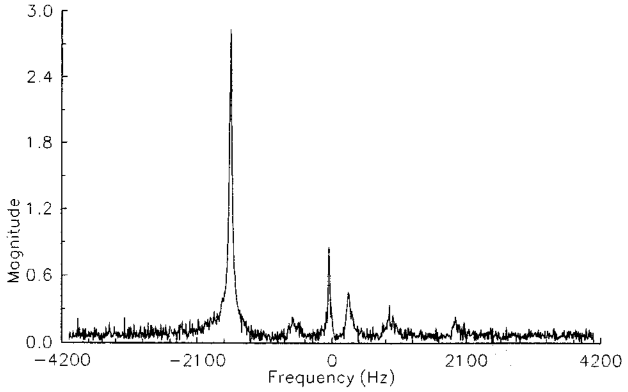

FIG. 6. Spectrum of a simulated FID with additive noise.

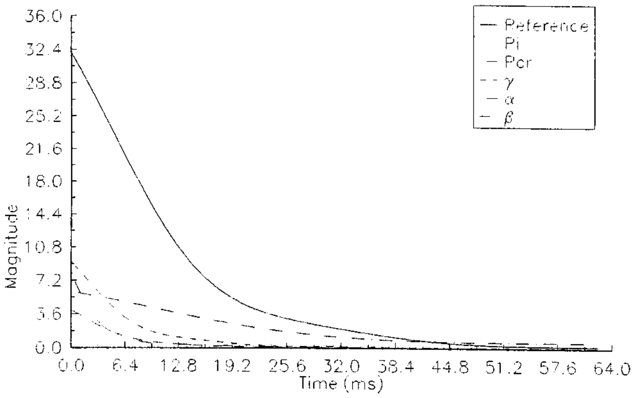

FIG. 7. Damping curves of the peaks of the simulated FID.

It is known that the magnitude of the FT of the signal

$$y(t) = \exp(-\alpha_y t), \quad t \geq 0, \quad \alpha_y > 0, \qquad [16]$$

; given by (Fig. 1a)

$$F_y(f) = \frac{1}{j2\pi f + \alpha_y}. \qquad [17]$$

is straightforward to see that

$$\lim_{f\to\infty} F_y(f) = \lim_{f\to-\infty} F_y(f) = 0.$$

For a practical application, $F_y(f)$ may be considered un-nportant for $f > |f_{95_y}|$, where

$$F_y(f_{95_y}) = 0.05 F_y(0). \qquad [18]$$

ı terms of α_y, f_{95_y} may be easily calculated to be

$$f_{95_y} \cong \frac{\alpha_y}{2\pi}. \qquad [19]$$

$$(\omega_{n+1} - \omega_n) > \alpha_y \quad \forall n \in [1, M), \qquad [20]$$

ıen Eq. [14] may be written as

$$Z_x(\tau) = \begin{cases} A_n \exp(-\alpha_n\tau)\exp(j\omega_n\tau)\exp(j\phi_n)F_n(\nu) \\ \quad \text{for}(\omega_n - \omega_{95_n}) \leq \nu \leq (\omega_n + \omega_{95_n}) \\ 0 \quad \text{elsewhere,} \end{cases} \qquad [21]$$

here F_n denotes the FT of the sequence in [15].

Condition [20] ensures that no overlapping occurs be-/een any of the individual F_n values ($1 \leq n \leq M$) and thus ıe may write

$$|Z_x(\tau)| = \sum_{n=1}^{M} A_n \exp(-\alpha_n\tau)|F_n(\nu)|. \qquad [22]$$

TABLE 2
Comparison of the Results for the Damping Factors Obtained by Various Techniques

Frequency (Hz)	Spectral fitting[a]	Time anaysis[a]	Wigner distribution[a]	Zak transform
−1590	11.2	11.3	11.0	11.0
−600	2.1	2.2	2.0	1.9
−60	22.3	21.8	19.3	19.9
240	4.7	5.4	5.9	6.2
860	3.5	3.7	3.4	3.1
1900	4.8	5.1	4.9	5.3

[a] Values reported in (*1*).

$|Z_x(\nu)|$ will have the same shape for all values of τ but different magnitude values, which will decline according to the damping law imposed by each of the α_n values appearing in Eq. [22]. This shape, described mathematically by Eqs. [17] and [21], is plotted in Fig. 1b for $n = 3$ and arbitrary values of A_n, ω_n, ϕ_n, and τ. It resembles the spectrum of the NMR signal. In the examples that will follow this point will become more evident.

A DISCRETE VERSION OF THE ZAK TRANSFORM

We begin by giving the definition of the discrete Fourier transform (DFT) (*4*), which will be used as a reference. The DFT $X[m]$ of a sequence $x[n]$ of length N is defined by

$$X[m] = \sum_{k=0}^{N-1} x[k]W_N^{mk}, \quad m \in [0, N), \qquad [23]$$

where $W_N = \exp(-j2\pi/N)$. The inverse transform is defined by

$$x[k] = \frac{1}{N}\sum_{m=0}^{N-1} X[m]W_N^{-mk}, \quad k \in [0, N). \qquad [24]$$

TABLE 3
Comparison of the Results for the Amplitudes Obtained by Various Techniques

Frequency (Hz)	Spectral fitting[a]	Time analysis[a]	Wigner distribution[a]	Zak transform
−1590	31,852	31,442	32,160	32,479
−600	11,015	10,106	10,695	10,215
−60	5,265	6,063	6,068	6,001
240	10,069	9,182	9,358	8,701
860	7,439	8,158	8,180	7,993
1900	3,377	4,096	3,843	3,858

[a] Values reported in (*1*).

ıhe above equations we have supposed that there are N ıpling points in the interval $[0, T_s]$ which produce N ectral values in the interval $[-F_s, F_s]$. The following fun- ımental relationship holds: $2F_s = N/T_s$. Thus, $x[k]$ must at the most F_s-bandlimited in frequency (Nyquist crite- ›n).

We define the discrete Zak transform (DZT) as

$$Z_x[\tau, m] = \sum_{k=0}^{N-1} x[\tau + k]W_N^{mk}, \quad \tau, m \in [0, N). \quad [25]$$

Note that because $x[\tau + k]$ is a shifted sequence of $x[k]$, will be F_s-bandlimited in frequency. Thus, doubling the inimum sampling frequency is not a prerequisite for the mputation of the DZT as it is for the computation of the WT. Finally, note that for $\tau = 0$, the DZT and the DFT e completely equivalent. Thus, $x[k]$ may be recovered from $[\tau, m]$ simply by setting $X[m]$ equal to $Z_x[0, m]$ in [24].

A windowed version of the DZT may also be defined,

$$[\tau, m] = \sum_{k=0}^{N-1} w[k]x[\tau + k]W_N^{mk}, \quad \tau, m \in [0, N), \quad [25]$$

ıere $w[\tau]$ denotes the window sequence. Windows are ighting functions applied to data to reduce spectral leakage sociated with finite observation intervals. Their use in the ımputation of the Zak transform is expected to smooth the sults in the time axes because the discontinuity at the ›undary of the periodic extension of the transformed se- ıence will be reduced. The examples in the next section rify this assertion.

EXAMPLES

As a first example, we examine a signal composed of the m of two damped complex sinusoids located at frequencies −600 and 200 Hz. Both of them have an amplitude of), a damping factor of 100 ms, and a phase of 0°. The signal supposed to have a positive time support. The sampling riod was chosen to be 391 μs; 128 data samples were com- ıted, which gives a total sampling time T_0 of 50 ms and a equency resolution of 20 Hz.

Figure 2 shows the magnitude of the DZT of this signal r a time interval of 16 ms and a frequency interval of 1280 z (i.e., a raster of 64 × 64 points). The magnitude of the 'DZT of the same signal is shown in Fig. 3 (same raster). ı this example, a Hamming window was used. It is given /

$$w[\tau] = 0.54 + 0.46 \cos(2\pi\tau/N),$$

$$\tau \in [-N/2, N/2], N = 64.$$

Figure 4 shows the DWT of the test signal. Note that, since the Wigner distribution of any signal is real-valued, the magnitude need not be calculated in this case. Observe the cross term which is located midway between the two frequencies present, i.e., at −200 Hz, and which does not appear in the Zak t–ω plane.

The next example considers a simulated phosphorus FID (distorted to explore a wider range of values of the parameters) composed of six peaks: a reference signal, P_i, PCr, γ, α, and β. Their frequencies, damping factors, amplitudes, and phases are given in Table 1. The same signal has been used in (*1*) for validation of the Wigner transform. One thousand twenty-four points were calculated using a sampling period of 122 μs.

Figure 5 shows its spectrum without noise. Figure 6 shows the spectrum of the same sequence corrupted by a Gaussian, white, additive sequence of 0 mean and 2000 variance. Figure 7 shows the damping curves that were calculated by the Zak transform. The estimated time constants and amplitudes are compared to those obtained by spectral fitting, time-domain analysis (optimized TLS method), and Wigner distribution (*1*). These values are given in Tables 2 and 3, respectively.

CONCLUSION

The Zak transform and its discrete form have been developed as an alternative time–frequency representation of damped sinusoids. It can be seen as a generalization of the discrete Fourier transform, in a way that time-variant signals and systems may also be described by it.

Interrelationships with a popular time–frequency representation, the Wigner distribution, were examined. The ZT proved not to exhibit the appearance of unwanted cross terms, as opposed to the WD. It is also computationally attractive, although a large amount of storage may be required to represent it.

The Zak transform may become a useful tool for the quantitative analysis of FIDs in NMR spectroscopy. A multidimensional Zak transform may also be defined as a generalization of the multidimensional Fourier transform. Its application to MRI will be discussed in a future paper.

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