

Nikolaos Ploskas & Nikolaos Samaras

# Linear Programming using MATLAB

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*To my family*  
– *Nikolaos Ploskas*

*To my son Stathis*  
– *Nikolaos Samaras*



# Preface

Linear Programming (LP) is a significant area in the field of operations research. The simplex algorithm is one of the top ten algorithms with the greatest influence in the 20th century and the most widely used method for solving linear programming problems (LPs). Since the introduction of the simplex algorithm in 1947, LP has been widely used in many practical problems. However, the size of practical LPs grew up significantly. Consequently, the simplex algorithm began to encounter computational issues in the solution of large LPs. A variety of methods have been proposed to strengthen the computational performance of simplex algorithm. Furthermore, new algorithms have been proposed to solve LPs, like the dual simplex algorithm, interior point methods, and exterior point simplex algorithms.

The main feature of this book is the presentation of a variety of LP algorithms and methods and especially the revised simplex method and its components. The computational performance of simplex algorithm on practical problems is usually far better than the theoretical worst case. This book includes the thorough theoretical and computational presentation of four LP algorithms:

- the revised primal simplex algorithm,
- the revised dual simplex algorithm,
- the exterior point simplex algorithm, and
- Mehrotra's interior point method.

Furthermore, this book presents:

- 11 presolve techniques,
- 11 scaling techniques,
- 6 pivoting rules, and
- 4 basis inverse and update methods.

The novelty of this book is that the presentation of each LP algorithm or method is focused on three aspects:

- Initially, the theoretical background is presented for each algorithm or method including their mathematical formulation.

- Secondly, a thorough numerical example is presented for each algorithm or method.
- Finally, a MATLAB code is given to fully cover the presentation of each algorithm or method. The MATLAB implementations that are presented in this book are sophisticated and allow the solution of large-scale benchmark LPs.

This book is addressed to students, scientists, and mathematical programmers. Students will learn various aspects about LP algorithms and especially the revised simplex algorithm through illustrative examples, while they can solve the examples using the MATLAB codes given in this book. This book covers thoroughly a course on Linear Programming whether MATLAB is used or not. Scientists and mathematical programmers will have a book in their library that presents many different components of simplex type methods, like presolve techniques, scaling techniques, pivoting rules, basis update methods, and sensitivity analysis. Moreover, the presentation of each component or algorithm is followed by a computational study on benchmark problems to present the computational behavior of the presented methods or algorithms.

Even though there is an abundance of books on LP, none of them comprehensively contains both the theoretical and practical backgrounds of each method. Our dual goal is to fill this gap and provide a book to students and researchers that can be used to understand many widely used LP methods and algorithms and find codes to implement their own algorithms. All codes presented in this book are sophisticated implementations of LP algorithms and methods, aiming towards a balance between speed and ease of use. Experienced programmers can further optimize the presented implementations to achieve better execution times. Moreover, we preferred not to present the simplex algorithm in tableau format, because it is not efficient in practice.

Our intention is to present many different LP methods and algorithms and especially present different computational techniques for the revised simplex algorithm. On the other hand, we decided not to include some other significant techniques, like LU basis update techniques and crash procedures for finding an initial basis. However, the inclusion of a part in this book presenting more advanced techniques is to be reconsidered in future editions.

We have selected to use MATLAB in order to implement the codes presented in this book for several reasons. First of all, MATLAB is a matrix language intended primarily for numerical computing. MATLAB is especially designed for matrix computations like, solving systems of linear equations or factoring matrices. Secondly, MATLAB gives us the ability to create concise codes. Hence, readers will focus on the implementation of the different steps of the algorithms and not on how linear algebra operations (e.g., matrix operations, decompositions, etc.) are implemented. Finally, MATLAB provides sophisticated LP algorithms and users can use them to solve their LPs. Readers that are not familiar with programming can also use MATLAB's graphical user interface for solving LPs.

We are thankful to MathWorks for providing us an academic license for MATLAB through their MathWorks Book Program. We also thank Charalambos Triantafyllidis, Themistoklis Glavelis, and the students of the department of Ap-

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*Nikolaos Ploskas*  
*Nikolaos Samaras*





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